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Some examples of non-existent combinators

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Dedicated to Corrado Böhm

Abstract

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We consider certain radical versions of familiar combinators. Some are shown not to exist while others present open problems.

We have all met interesting combinators. Many elementary ones appear in Ray Smullyan's *To Mock a Mockingbird* [4]. These include the fixed point combinators often called "paradoxical". Less elementary examples occur in more advanced texts such as Curry and Feys [3], and Barendregt [1]. These include universal generators, Plotkin terms, recurrent combinators and easy terms.

In all these cases it is of great interest to see what additional conditions can and cannot be imposed on such combinators. We are here particularly interested in learning how to prove that certain radical versions of these combinators do not exist.

In several cases we can do this only for combinatory logic and not for lambda calculus. In this note we shall concentrate on open problems which interest us.

1. Fixed point combinators

(a) (Böhm, see [1, p. 142]) Y is a fixed point combinator $\Leftrightarrow$ $SIY=Y$. If Y is a fixed point combinator then so is $Y(SI)$. Is there a fixed point combinator Y such that $Y=Y(SI)$?

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Remark. None exists with $Y \twoheadrightarrow Y(\lambda ab.b(ab))$ because such a combinator must be of order zero.

(b) (Smullyan) Recall that the mockingbird M has the reduction rule

$$Mx \to xx$$

(ω in [1]). Is there a B,M combination which is a fixed point combinator? This problem has been studied by Wos and McCune [9]. It is delicate in that $M(BxM)$ is always a fixed point of x. Note that $BM(B(BM)B)$ has the right Böhm tree to be a fixed point combinator but it is not one.

Remark. There is no B,M combination Y such that $Yx \twoheadrightarrow x(Yx)$. Our proof uses simply typed lambda calculus! Wos and McCune have also obtained this result by a different method. Here we give the proof.

We begin by some simple observations.

Lemma 1. *Suppose that F is a B,M combination and $Fx(1)\ldots x(n) \twoheadrightarrow x(1)X(1)\ldots X(m)$. Then F has a normal form.*

Proof. The proof uses the standardization theorem for B,M combinations, by inducting on the length of a head reduction. □

Now suppose that $Fx \twoheadrightarrow X$ and F is a normal B,M combination.

Lemma 2. If *U is a subterm of X which contains x then x is the rightmost symbol in U.*

Lemma 3. *If U is a subterm of X which is a redex then U contains x.*

Lemma 4. *If $X \to Z$ and Z contains the subterm UV such that U is a trace of a subterm of X not in function position then V contains x.*

The proof of Lemmas 2 and 3 is by induction on the length of a reduction sequence. Lemma 4 is proved by case analysis. Finally observe that an occurrence of B in F can have zero, one, or two arguments.

We shall now type Fx, X, and their subterms in the free Cartesian closed type structure over a single ground domain 0. We shall begin to write [ ] for typed application using ( ) for the binary (untyped) operation of type $0 \times 0 \to 0$. We replace occurrences of B with zero arguments by the constant $b: 0 \to 0$. We replace occurrences of B with one argument as follows

$$Bf \to [Bf]$$

where the second occurrence of B has $B:(0\to 0)\to(0\to 0)$. In addition, we replace occurrences of B with two arguments as follows

$$Bfg \to [*\langle f, g\rangle]$$

where $*:(0\to 0)\times(0\to 0)\to(0\to 0)$. Finally, occurrences of ( ) are replaced as follows

$$(UV)\to[(\,)\langle U, V\rangle]$$

and $x:0$. Of course we shall use the obvious abbreviations in using ( ) and $*$.

We have now the following reduction rules

$$(([bx]y)z) \to (x(yz)),$$

$$([[Bf]x]y) \to [f(xy)],$$

$$[f*g\,x] \to [f[gx]],$$

$$[Mx] \to (xx).$$

Lemma 5. *X and its subterms inherit a typing from Fx as $Fx \twoheadrightarrow X$. Moreover, the typing of X is unique.*

Proof. By induction on the length of a reduction. At the same time verify that a subterm of X has type 0 iff it is open. b appears only with no arguments, or a sequence of open ones. B appears only with one closed argument followed by a sequence of open ones. Finally, $*$ appears only followed by two closed arguments followed by a sequence of open ones. Thus the replacements of B by b, B, and $*$ are determined by X and so is its typing.

Now suppose that there is a B,M combination Y such that

$$Yx \twoheadrightarrow x(Yx).$$

By Lemma 1 we can assume that Y is normal and so by Lemma 5 we have a typed reduction

$$[Yx]\twoheadrightarrow(x[Yx]).$$

Consider the last time the outermost symbol changes from [ ] to ( ). We have

$$[Yx]\twoheadrightarrow[MX]\to(XX)\twoheadrightarrow(x[Yx]).$$

Now X contains x and so by Lemma 2 it is the rightmost leaf of X. Moreover this is true for any reduct of X. Now any reduction $(UV)\to(WZ)$ transfers the rightmost leaf of U to Z. Since $[Yx]$ has only one occurrence of x, (XX) cannot reduce to a redex. But now we have a contradiction since $x\neq[Yx]$. $\square$

2. **Recurrent combinators** (Venturini-Zilli)

Recall that M is recurrent if whenever $N=M$ we have $N \twoheadrightarrow M$. Let us say that M is hyperrecurrent if whenever $N=M$ we have N recurrent. Equivalently M is hyperrecurrent if whenever $P=M=Q$ we have $P \twoheadrightarrow Q$ and $Q \twoheadrightarrow P$. Are there any hyperrecurrent combinators? The problem comes up immediately when the Ershov–Visser theory for $=$ is applied to $\twoheadrightarrow$ ([8]).

Remark. Hyperrecurrent combinators do not exist for combinatory logic ([6]). The proof uses the following. Say that M has upward Church–Rosser if whenever $P \twoheadrightarrow M \twoheadleftarrow Q$ there exists an N such that $P \twoheadleftarrow N \twoheadrightarrow Q$.

Theorem ([6]). *M has upward Church–Rosser $\Leftrightarrow$ M is an atom.*

3. **Universal generators** (Barendregt)

Remember that M is a universal generator if for each combinator P there exists a superterm Q of P such that $M \twoheadrightarrow Q$. Say that M is a uniform universal generator if there exists a context $C[\cdot]$ such that for each combinator P, $M \twoheadrightarrow C[P]$. Is there a uniform universal generator?

Remark. For combinatory logic, if we restrict the context $C[\cdot]$ to be of the form $(N\ \cdot)$ then there is no such term. This special case actually comes up in another way outlined below. First, we would like to give the proof, since it is not too long.

The proof proceeds by contradiction.

Suppose that P and Q are given such that for all R

$$P \twoheadrightarrow QR.$$

Let O be an infinite set of applicative combinations of Ω

$$SII(SII)$$

such that for any U and V in O if U is a subterm of V then $U=V$. There are many choices for O. By the standardization theorem there exists an enumeration of O possibly with repetition $A_1, A_2, \ldots, A_n, \ldots$ such that the following reduction diagram exists

$$P \underset{\text{head}}{\twoheadrightarrow} Q_0B_0 \underset{\text{head}}{\twoheadrightarrow} Q_1B_1 \underset{\text{head}}{\twoheadrightarrow} \cdots \underset{\text{head}}{\twoheadrightarrow} Q_nB_n \underset{\text{head}}{\twoheadrightarrow} \cdots.$$

where $[Q_iB_i \underset{\text{internal}}{\twoheadrightarrow} QA_i]$.

Find an enumeration of O possibly with repetition $C_0, C_1, \ldots, C_n, \ldots$ such that the following reduction diagram exists

$$P \underset{\text{head}}{\twoheadrightarrow} P_0 D_0 \underset{\text{head}}{\twoheadrightarrow} P_1 D_1 \underset{\text{head}}{\twoheadrightarrow} \cdots \underset{\text{head}}{\twoheadrightarrow} P_n D_n \underset{\text{head}}{\twoheadrightarrow} \cdots$$

where $D_i \twoheadrightarrow C_i$ and every head reduct of P whose last component reduces to a member of O is explicitly listed. Of course, among the D_i are the B_j. In addition, for each n find an enumeration of a subset of O possibly with repetition $E_{n0}, E_{n1}, \ldots, E_{nm}, \ldots$ such that the following reduction diagram exists

$$QA_n \underset{\text{head}}{\twoheadrightarrow} Q_{n0} F_{n0} \underset{\text{head}}{\twoheadrightarrow} Q_{n1} F_{n1} \underset{\text{head}}{\twoheadrightarrow} \cdots \underset{\text{head}}{\twoheadrightarrow} Q_{nm} F_{nm} \underset{\text{head}}{\twoheadrightarrow} \cdots$$

where $F_{nm} \twoheadrightarrow E_{nm}$ and every head reduct of QA_n whose last component reduces to a member of O is explicitly listed. We have made it appear that the enumeration $E_{n0}, E_{n1}, \ldots, E_{nm}, \ldots$ is infinite. This will be shown below. We shall now compare the reduction sequences constructed above.

Quite generally, suppose that we are given

$$J_1 L_1 \underset{\text{head}}{\twoheadrightarrow} J_2 L_2 \underset{\text{head}}{\twoheadrightarrow} \cdots \underset{\text{head}}{\twoheadrightarrow} J_m L_m \underset{\text{head}}{\twoheadrightarrow} \cdots$$

and

$$J_1 L_1 \underset{\text{int.}}{\twoheadrightarrow} T_1 \underset{\text{head}}{\twoheadrightarrow} T_2 \underset{\text{head}}{\twoheadrightarrow} \cdots \underset{\text{head}}{\twoheadrightarrow} T_m \underset{\text{head}}{\twoheadrightarrow} \cdots$$

where the first reduction sequence is infinite. We wish to compare the first with the second. First assume that the internal reduction is one step i.e.

$$J_1 L_1 \xrightarrow[\text{int.}]{R} T_1$$

with redex R. If $J_m L_m \twoheadrightarrow T_n$ by a complete reduction of all the residuals of R in $J_m L_m$ then so does $J_{m+1} L_{m+1} \twoheadrightarrow T_{n+1}$ by a complete reduction of all residuals of R except when some residual of R sits at the head of $J_m L_m$. In the latter case $J_{m+1} L_{m+1} \twoheadrightarrow T_n$ by a complete reduction of all residuals of R. Thus there is a monotone increasing function f which maps the positive integers onto one of its initial segments such that

$$J_m L_m \twoheadrightarrow Tf(m)$$

by a complete reduction of all the residuals of R in $J_m L_m$. Thus since residuals of R are disjoint and $J_m L_m$ can coincide with a residual of R at most once, for all but one m we have $Tf(m)$ of the form $Mf(m)\, Nf(m)$ with $J_m \twoheadrightarrow Mf(m)$ and $L_m \twoheadrightarrow Nf(m)$. In particular, if there are infinitely many noninterconvertible L_m there are infinitely many noninterconvertible N_j and the head reduction sequence beginning with T_1 is infinite. Now we return to the general case assuming that there are infinitely many noninterconvertible L_m. Thus there is a monotone increasing function g mapping the positive integers onto one of its initial segments such that $J_m L_m \twoheadrightarrow Tg(m)$ and for all but finitely many m $Tg(m)$ has the form $Mg(m)Ng(m)$ with $J_m \twoheadrightarrow Mg(m)$ and $L_m \twoheadrightarrow Ng(m)$.

Let us now consider the previous enumerations of subsets of O, possibly with repetition, as infinite words on the alphabet O. Given two infinite words W_0 and W_1 we say that W_0 can be obtained from W_1 by deletions and contractions if indeed W_0 can result from finitely deletions of letters from W_1 followed by possibly infinitely many parallel contractions of adjacent identical letters

$$a^n \to a$$

As said above B_n is one of the D_j, say D_m, so the infinite word

$$A_n\ E_{n0}\ E_{n1} \dots E_{nk} \dots$$

can be obtained from the infinite word

$$C_m\ C_{m+1}\, C_{m+2} \dots C_{m+k} \dots$$

by deletion and contraction according to the previous paragraphs.

Consider our head reduction beginning with QA_n. In any of the subreductions $QA_n \twoheadrightarrow Q_{nk}F_{nk}$ either all the residuals of A_n in F_{nk} are projected when $F_{nk} \twoheadrightarrow E_{nk}$ or E_{nk} coincides with one of them. This E_{n0} is by the choice of O. Thus the infinite words

$$A_n\ E_{n0}\ E_{n1} \dots E_{nk} \dots$$

$$A_m\ E_{m0}\ E_{m1} \dots E_{mk} \dots$$

can differ only at entries which are occurrences of A_n or A_m.

Now the infinite word

$$A_0\ E_{00}\ E_{01} \dots E_{0k} \dots$$

is obtained from

$$C_p\ C_{p+1} \dots C_{p+k} \dots$$

for some p by deletion and contraction. Let q be so large that if A_q is C_r, then

(a) all deletions are of C_i for $i<r$,

(b) there are at least two noninterconvertible C_i with $p<i<r$ that are not deleted and not interconvertible with either C_p or C_r.

Let W_0, W_1, resp. W_2 be the infinite words obtained from

$$A_0\, E_{01}\, E_{02} \dots E_{0k} \dots$$

$$C_p\, C_{p+1} \dots C_{p+k} \dots$$

$$C_r\ C_{r+1} \dots C_{r+k} \dots$$

by deleting all occurrences of A_0 and A_q. We have that

(1) $W_1 = xW_2$ for some finite word x;

(2) W_0 results from both W_1 and W_2 by deletion and contraction;

(3) W_1, thus W_0 and W_2, has infinitely many distinct letters at least two of which are not deleted from x;

(4) the first letter of W_2 lies to the right of the last deletion in going from W_1 to W_0.

We shall now look more closely at the process of deletion and contraction of infinite words. Let W be an infinite word. Adjacent letters in W are said to be equivalent if they are instances of the same letter. Equivalence generates an equivalence relation which partitions the letters of W into consecutive instances of the same letter. These are called the blocks of W. When W contains infinitely many letters the blocks of W are finite words. Let W^* result from W by replacing each block of W by a single instance of the letter of the block. When W has infinitely many letters W^* results from W by deletions and contractions.

Returning to the above, the block of W_2 containing the first letter after the last deletion in the passage from W_2 to W_0 will be called the boundary block. Consider any block B strictly to the right of the boundary block. B yields an entire block of W_0 under the deletion and contraction of W_1 and another entire block of W_0 under the deletion and contraction of W_2. Moreover, no other one can yield the same block of W_0 in either case. Let W_3 be the part of W_2 strictly to the right of the boundary block. Thus the deletions and contractions above induce isomorphisms from W_3^* onto final segments of W_0^*. These isomorphisms must be identical for otherwise W_3^* is periodic contradicting the fact that W_1 has infinitely many letters. In particular, there is only one final segment of W_0^* in question. Let this be W_4^*, where $W_0 = yW_4$. Also, let $W_2 = zW_3$. We see that xz yields y by deletion and contraction in the deletion and contraction of W_1 to W_0, and z yields y by deletion and contraction in the deletion and contraction of W_2 to W_0. Note that in the first deletion and contraction nothing is deleted from z.

Finally we may suppose that we have finite words x, y, and z such that xz yields y by deletion and contraction with no deletions in z, and z yields y by deletions and contractions. If we make the deletions in x first we obtain a word u such that uz yields y by contractions and z yields y by deletions and contractions. Hence z yields uz by deletions and expansions. Since z is a suffix of $uz = a_1 a_2 \ldots a_r$ this is equivalent to there being $i_1 < i_2 < \cdots < i_k$ and $e(i_1) \ldots e(i_k)$ such that

$$a_1 a_2 \ldots a_r = a_{i_1}^{e(i_1)} \ldots a_{i_k}^{e(i_k)}$$

where without loss of generality we can assume a_{i_j} and $a_{i_{j+1}}$ are different letters. But by the pigeonhole principle we have $i_1 < e(i_1) + 1$ and all the letters of u are the same a_{i_1}. This contradicts the choice of p and q.

The special case context comes up in the following way. Recall that Plotkin found P and Q such that for all combinators M we have $PM = Q$ but $P \neq KQ$ ([1, p. 455]). Later we proved that every recursively enumerable β-closed set has the form

$$\{M : PM = Q\}$$

([5]). Which sets have the form

$$\{M : Q \twoheadrightarrow PM\}?$$

4. Easy terms (Jacopini)

Recall that M is easy if for each combinator N the equation $N=M$ is consistent with β-conversion ([1, p. 402]). Say that M is hypereasy if whenever the equation $Fx=Gx$ is solvable in some model of β-conversion the equation $FM=GM$ is consistent. Is there a hypereasy combinator?

Answer. No. Barendregt and I noticed that if P is a Plotkin combinator, i.e. for all M we have $PM=P$ but $Px\neq P$ then for

$$F=\lambda x\cdot\langle xK(P(xK^*)), xKP\rangle,$$

$$G=\lambda x\cdot\langle K^*, K\rangle$$

the equation $FM=GM$ is not consistent for any combinator M. However, the equation $Fx=Gx$ is consistent. To see this let a and b be new constants and consider the reductions

$$aX\to K^* \quad \text{if } Pb\twoheadrightarrow X,$$

$$aX\to K \quad \text{if } P\twoheadrightarrow X.$$

These reductions satisfy the conditions of Mitschke's theorem ([1, p. 401]) and are thus Church–Rosser. Hence the congruence generated by the reductions is consistent and in the resulting term model $\langle a, b\rangle$ is a solution to $Fx=Gx$.

Let us say that M is n-easy if the shortest equational proof that M is inconsistent with anything has at least n lines. Whatever this means an easy term is n-easy for all n. Are there n-easy terms which are not $n+1$-easy? The answer to this, somewhat surprisingly, is yes for infinitely many n. The proof is too long to include here.

For some new deep and beautiful results about easy terms the reader should consult the paper by Berarducci and Intrigilia [2] in this volume.

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